1. *(a) Show that* $\mathbf{C}^*$ *and* Δ^* *are not biholomorphic;*
   (b) Show that Δ^* *is not biholomorphic to an annulus* $A = \{z \in \mathbf{C} \mid 0 < r < |z| < 1\}$.

If you are curious, a proof that two annuli $A_{r,1} = \{0 < r < |z| < 1\}$ *and* $A_{s,1} = \{0 < s < |z| < 1\}$ *are biholomorphic if and only if* $r = s$ *can be found in Greene-Krantz, Thm.7.9.1, p.240.*

Sol.: (a) Suppose that there exists a biholomorphism $f\colon \mathbf{C}^* \to \Delta^*$. Then $f \circ \exp\colon \mathbf{C} \to \Delta^*$ would define a bounded entire holomorphic map. This is impossible by Liouville's theorem.

(b) Suppose that there exists a bihomolorphism $f\colon \Delta^* \to A$. By the same arguments used in the previous case, $z = 0$ is necessarily a removable singularity for f and f extends to a holomorphic map $\tilde{f}\colon \Delta \to A$. On the other hand, by topological reasons (Δ is simply connected, while A is not), the image $\tilde{f}(\Delta)$ is properly contained in A. Contradiction.

Alternatively one could use the fact that the automorphism groups of $Aut(\mathbf{C}^*)$, $Aut(\Delta^*)$ and $Aut(A)$ are non-isomorphic.

- $Aut(\mathbf{C}^*) = \{z \mapsto az,\ a \in \mathbf{C}^*\} \cup \{z \mapsto a/z,\ a \in \mathbf{C}^*\}$.

Let $f\colon \mathbf{C}^* \to \mathbf{C}^*$ be a biholomorphism. Then neither $z = 0$ nor ∞ can be essential singularities of f, otherwise injectivity would fail by Casorati-Weierstrass' theorem.
If $z = 0$ is a removable singularity, then $0 \mapsto 0$ and f extends to an automorphism of $\mathbf{C}$. Namely, it is a degree 1 polynomial of the form $f(z) = az$, with $a \in \mathbf{C}^*$.
If $z = 0$ is a pole, then $\lim_{z\to 0} |f(z)| = \infty$. In this case $1/f$ extends to an automorphism on $\mathbf{C}$ mapping 0 to 0. Namely $\frac{1}{f(z)} = az$, for $a \in \mathbf{C}^*$. Equivalently $f(z) = a/z$, for $a \in \mathbf{C}^*$.

Alternatively: Since f has no essential singularities, then either 0 and ∞ are both removable singularities or they are both poles. In the first case, f extends to an automorphism of the Riemann sphere $\tilde{f}\colon S^2 \to S^2$ mapping $0 \mapsto 0$ and $\infty \mapsto \infty$. Namely $f(z) = az$, with $a \in \mathbf{C}^*$. In the second case, f extends to an automorphism of the Riemann sphere mapping $0 \mapsto \infty$ and $\infty \mapsto 0$. Namely, $f(z) = a/z$, with $a \in \mathbf{C}^*$.

- $Aut(\Delta^*) = \{z \mapsto e^{i\theta} z,\ \theta \in \mathbf{R}\}$.

Let $f\colon \Delta^* \to \Delta^*$ be a biholomorphism. Then $z = 0$ is necessarily a removable singularity of f and f extends to an automorphism of Δ, mapping $0 \mapsto 0$. Hence $f(z) = e^{i\theta} z$, with $\theta \in \mathbf{R}$.

- $Aut(A)$ contains $\{z \mapsto e^{i\theta} z,\ \theta \in \mathbf{R}\} \cup \{z \mapsto e^{i\theta} r/z\}$.

Infinite series of holomorphic and meromorphic functions.

2. *Let* D *be a domain in* $\mathbf{C}$. *Prove that if the series of holomorphic functions* $\sum_k g_k(z)$ *converges normally on* $A \subset D$, *then it converges uniformly on* $A \subset D$.

Sol.: By assumption the series converges normally, i.e. $\sum_k \|g_k\|_A < \infty$, where $\|g_k\|_A = \sup_{z\in A} |g_k(z)|$. Then for all $z \in A$ and $N > M$ one has

$$|\sum_{k=1}^{N} g_k(z) - \sum_{k=1}^{M} g_k(z)| \le \sum_{k=M}^{N} |g_k(z)| \le \sum_{k=M}^{N} \sup_{z\in A} |g_k(z)| \le \sum_{k=M}^{N} \|g_k\|_A \to 0, \quad \text{for } N, M \to \infty.$$

Hence the sequence of the partial sums of the series is uniformly Cauchy and the series converges uniformly on $A \subset D$.

3. *Show that the series* $\sum_{n\ge 1} \frac{z^n}{n^2}$ *converges uniformly on the unit disc.*

Sol.: For all $z \in \Delta$ one has
$$\left|\frac{z^n}{n^2}\right| \leq \frac{1}{n^2}.$$
Hence
$$\sum_{n\geq 1}\left|\frac{z^n}{n^2}\right| \leq \sum_{n\geq 1}\frac{1}{n^2} < \infty,$$
which implies that the series $\sum_{n\geq 1}\frac{z^n}{n^2}$ converges absolutely and uniformly on Δ.

4. *(a) Show that* $\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} kz^{k-1}$, *for* $z \in \Delta$, *and that* $4 = \sum_k k\frac{1}{2^{k-1}}$.
   (b) Show that $\log(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k}$, *for* $z \in \Delta$.

Sol.: (a) We know that the geometric series $\sum_{k\geq 0} z^k$ converges uniformly on compact sets in Δ to the sum $\frac{1}{1-z}$. Then, by Weierstarss theorem, the uniform holds true also for the derivatives:
$$\left(\sum_{k\geq 0} z^k\right)' = \sum_{k\geq 1} kz^{k-1} \longrightarrow \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}.$$
In particular, for $z = \frac{1}{2}$, we obtain the identity
$$\sum_{k\geq 1} k\frac{1}{2^{k-1}} = \frac{1}{(1-\frac{1}{2})^2} = 4.$$

(b) By Exercises 5 and 6 in Sheet 3, the uniform convergence holds true for the primitives with value 0 at $z = 0$:
$$\int\frac{1}{1-z} = -\log(1-z) = \int\sum_{k\geq 0} z^k = \sum_{k=1}^{\infty}\frac{z^k}{k}$$
and
$$\log(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k}.$$

5. *Show that* $\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}$ *defines a meromorphic function on* $\mathbf{C}$. *Determine its poles and their orders.*

Sol.: The n^{th} term of the series $f_n(z) = \frac{1}{(z+n)^2}$ is a meromorphic function on $\mathbf{C}$ with a pole of order 2 at $z = -n$.
Fix $R > 0$. Then for $n > R$ the functions f_n are holomorphic on $\{z \in \mathbf{C} \ : \ |z| \leq R\}$. Since
$$\left|\frac{1}{(z+n)^2}\right| \leq \frac{1}{|n-|z||^2} \leq \frac{1}{|n-R|^2},$$
the series
$$\left|\sum_{n=R}^{\infty}\frac{1}{(z+n)^2}\right| \leq \sum_{n=R}^{\infty}\frac{1}{|n-R|^2}$$

converges uniformly to a holomorphic function on $D(0,R)$ and $\sum_{n\geq 1}^{\infty} f_n(z)$ defines a meromorphic function therein. By taking R larger and larger, we conclude that the original series converges locally uniformly and defines a meromorphic function on $\mathbf{C}$ with poles of order 2 at all integers $n \geq 1$. This last statement follows from the fact that the poles of the functions f_n are pairwise disjoint.

6. *Show that* $\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ *defines a meromorphic function on* $\mathbf{C}$*. Determine its poles and their orders.*

Sol.: The n^{th} term of the series $f_n(z) = \frac{1}{z^2+n^2}$ is a meromorphic function on $\mathbf{C}$ with poles of order 1 at $\pm in$, with residue $\frac{1}{2in}$ and $\frac{-1}{2in}$, respectively.
Fix $R > 0$. Then for all $n > R$, the functions f_n are holomorphic on $D(0,R)$. In addition, one has

$$\left|\frac{1}{z^2+n^2}\right| \leq \frac{1}{|R^2-n^2|},$$

which implies that the series $\sum_{n>R} \frac{1}{z^2+n^2}$ converges uniformly to a holomorphic function on $D(0,R)$. It follows that the series $\sum_{n\geq 1} \frac{1}{z^2+n^2}$ converges uniformly to a meromorphic function on $D(0,R)$. By taking R larger and larger, we conclude that the original series converges locally uniformly and defines a meromorphic function on $\mathbf{C}$ with poles of order 1 at $\pm in$, for all $n \geq 1$. This last statement follows from the fact that the poles of the functions f_n are pairwise disjoint.

7. *Show that* $g(z) = \left(\frac{\pi}{\sin \pi z}\right)^2$ *converges uniformly to 0 for* $|Im(z)| \to \infty$*.*

Sol.: For $z = x + iy$, one has the following estimate, independent of x:

$$\left|\frac{\pi}{\sin \pi z}\right|^2 = \frac{\pi^2}{\sin^2 \pi x \cosh^2 \pi y + \cos^2 \pi x \sinh^2 \pi y} = \frac{\pi^2}{\sin^2 \pi x + \sinh^2 \pi y} \leq \frac{\pi^2}{\sinh^2 \pi y}.$$

Hence

$$\lim_{|y|\to\infty} \left|\frac{\pi}{\sin \pi z}\right|^2 = 0,$$

uniformly with respect to x.

8. *Compute* $\lim_{z\to 0} \left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2}$ *and* $\lim_{z\to 0} \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2}$*. Deduce that* $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$*.*

Sol.: From the Taylor expansion $\sin \pi z = \pi z - \frac{1}{6}\pi^3 z^3 + \ldots$ we obtain

$$\left(\frac{\pi}{\sin \pi z}\right)^2 = \left(\frac{\pi}{\pi z - \frac{1}{6}\pi^3 z^3 + \ldots}\right)^2 = \frac{1}{z^2}\left(\frac{1}{1-\frac{1}{6}\pi^2 z^2 + \ldots}\right)^2 = \frac{1}{z^2} + \frac{\pi^2}{3} + z^2(\ldots)$$

and

$$\lim_{z\to 0} \left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2} = \frac{\pi^2}{3}.$$

On the other hand, from the identity $\left(\frac{\pi}{\sin \pi z}\right)^2 = \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2}$ and the fact that the function $\sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2}$ is holomorphic around 0, we deduce

$$\lim_{z\to 0} \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2} = 2 \lim_{z\to 0} \sum_{n\geq 1} \frac{1}{(z-n)^2} = \sum_{n\geq 1} \frac{1}{n^2}.$$

Conclusion:
$$\frac{\pi^2}{3} = 2\sum_{n\geq 1}\frac{1}{n^2} \quad \text{and} \quad \sum_{n\geq 1}\frac{1}{n^2} = \pi^2/6.$$

9. *Show that* $\sum_{n\in\mathbf{Z}}\frac{1}{(z^3-n^3)}$ *converges normally to a meromorphic function. Locate the poles and find the corresponding principal parts of the function.*

Sol.: For $n \in \mathbf{Z}_{\neq 0}$, the n^{th} term of the series $f_n(z) = \frac{1}{z^3-n^3}$ is a meromorphic function on $\mathbf{C}$ with poles of order 1 at n, $n\alpha$, $n\bar{\alpha}$, where α is a third root of unity, and principal parts given by

$$\frac{a}{z-n}, \quad \frac{b}{z-n\alpha}, \quad \frac{c}{z-n\bar{\alpha}}, \qquad \text{for some } a,\, b,\, c \in \mathbf{C},$$

respectively. For $n = 0$, the function $f_0(z) = \frac{1}{z^3}$ has a pole of order 3 at $z = 0$, with principal part equal to $\frac{1}{z^3}$. Now the arguments are similar to the ones used in Exercises 5 and 6. Fix $R > 0$. Then for all $|n| > R$, the functions f_n are holomorphic on $D(0,R)$ and on $D(0,R)$ the following estimate holds
$$\left|\frac{1}{z^3-n^3}\right| \leq \frac{1}{||n|^3-R^3|}.$$
Hence the series $\sum_{|n|>R}\frac{1}{z^3-n^3}$ converges uniformly to a holomorphic function on $D(0,R)$ and the series $\sum_{n\in\mathbf{Z}}\frac{1}{z^3-n^3}$ converges uniformly to a meromorphic function on $D(0,R)$. We conclude that the original series converges locally uniformly and defines a meromorphic function f on $\mathbf{C}$ with poles of order 1 at n, $n\alpha$, $n\bar{\alpha}$, for all $n \neq 0$ and a pole of order 3 at $z = 0$. The principal parts of f at the poles coincide with the principal parts of the functions f_n.

10. *Find a meromorphic function* $f\colon \mathbf{C} \to \mathbf{C}$ *with simple poles at the positive integers with residue 1.*

Sol.: We construct f as the sum of a converging series of meromorphic functions f_n, for $n \geq 1$, each with a simple pole of residue 1 at n. Since the series $\sum_{n\geq 1}\frac{1}{z-n}$ does not converge, we take

$$\sum_{n\geq 1} f_n(z), \qquad \text{with } f_n(z) = \frac{1}{z-n} + \frac{1}{n}.$$

Next we show that the above series converges locally uniformly on $\mathbf{C}$ to a meromorphic function with the required properties.
Fix $R > 0$. For all $n > R$, the functions f_n are holomorphic on the disk $D(0,R)$. Moreover, on $D(0,R)$ one has
$$\left|\frac{z}{n(z-n)}\right| \leq \left|\frac{R}{n^2-nR}\right|,$$
which implies that $\sum_{n>R} f_n(z)$ converges uniformly to a holomorphic function on $D(0,R)$ and $\sum_{n\geq 1} f_n(z)$ converges uniformly to a meromorphic function on $D(0,R)$. By taking R larger and larger, we conclude that the original series defines a meromorphic function on $\mathbf{C}$ with poles of order 1 and residue 1 at all integers $n \geq 1$.

Infinite products of holomorphic functions.

11. *Prove that the infinite product* $\prod_j a_j$*, where* $a_j = \frac{1}{2}$ *for every* j*, does not converge.*

Sol.: The partial products $\prod_{j=1}^n \frac{1}{2} = \frac{1}{2^n}$ tend to 0, for $n \to \infty$. This contradicts the definition of a converging infinite product, for which the above limit of the partial products has to be $\neq 0$.

12. *Prove that* $\prod_{n=2}^\infty \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

Sol.: One has

$$\prod_{n=2}^N \left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^N \frac{n+1}{n} \cdot \prod_{n=2}^N \frac{n-1}{n} = \frac{3}{2}\frac{4}{3} \cdots \frac{N}{N-1}\frac{N+1}{N} \cdot \frac{1}{2}\frac{2}{3} \cdots \frac{N-1}{N}$$

$$= \frac{1}{2}\frac{N+1}{N} \to \frac{1}{2}, \qquad \text{for } N \to \infty.$$

13. *Prove that for* $|z| < 1$ *one has*

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots = \frac{1}{1-z}.$$

Sol.: One has

$$(1+z)(1+z^2) = 1+z+z^2+z^3, \quad (1+z)(1+z^2)(1+z^4) = 1+z+z^2+z^3+z^4+z^5+z^6+z^7, \ldots$$

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots(1+z^{2^k}) = 1+z+z^2+z^3+z^4+\ldots+z^{1+2+\ldots+2^k}.$$

Hence for $k \to \infty$ we get the sum of the geometric series

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots(1+z^{2^k}) \longrightarrow \frac{1}{1-z}.$$

14. *Prove that*

$$\prod_{n=1}^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

Sol.: Let $a_n = -n$ and $p_n \equiv 1$. Since

$$\sum_{n=1}^\infty \left(\frac{R}{n}\right)^2 < \infty \quad \forall R > 0,$$

by the convergence criterion of Thm.8.2.2 in Greene-Krantz, the infinite product

$$\prod_{n=1}^\infty (1 + \frac{z}{n}) e^{z/n}$$

converges absolutely and uniformly on all $\mathbf{C}$. It defines an entire holomorphic function with simple zeros at negative integers $n \in \mathbf{Z}_{<0}$, and no other zeros.

15. *Construct an entire function on $\mathbf{C}$ with simple zeros at n^2, for $n \geq 0$, and no other zeros.*

Sol.: Let $a_n = n^2$ and $p_n \equiv 0$. Since

$$\sum_{n=1}^{\infty} \frac{R}{n^2} < \infty, \quad \forall R > 0,$$

by the convergence criterion of Thm.8.2.2 in Greene-Krantz, the infinite product

$$\prod_{n=1}^{\infty} (1 - \frac{z}{n^2})$$

converges absolutely and uniformly on all $\mathbf{C}$. Then

$$z \prod_{n=1}^{\infty} (1 - \frac{z}{n^2})$$

defines an entire holomorphic function with simple zeros at n^2, for $n \geq 0$, and no other zeros.

16. *Construct an entire function that has simple zeros on the positive real axis at the points $\sqrt{n}$, for $n \geq 1$, double zeros on the imaginary axis at the points $\pm i\sqrt{n}$, for $n \geq 1$, and no other zeros.*

Sol.: An entire function that has simple zeros on the positive real axis at the points $\sqrt{n}$, for $n \geq 1$ is given by

$$f(z) = \prod_{n \geq 1} \left(1 - \frac{z}{\sqrt{n}}\right) e^{\frac{z}{\sqrt{n}} + \frac{1}{2}(\frac{z}{\sqrt{n}})^2 + \frac{1}{3}(\frac{z}{\sqrt{n}})^3}.$$

This can be seen by applying the convergence criterion of Thm.8.2.2 in Greene-Krantz with $a_n = \sqrt{n}$ and $p_n = 3$ for all n.

An entire function that has double zeros on the imaginary axis at the points $\pm i\sqrt{n}$, for $n \geq 1$ is given by

$$g(z) = \prod_{n \geq 1} \left(\left(1 - \frac{z}{i\sqrt{n}}\right) e^{\frac{z}{i\sqrt{n}} + \frac{1}{2}(\frac{z}{i\sqrt{n}})^2 + \frac{1}{3}(\frac{z}{i\sqrt{n}})^3} \right)^2 \left(\left(1 + \frac{z}{i\sqrt{n}}\right) e^{-(\frac{z}{i\sqrt{n}} + \frac{1}{2}(\frac{z}{i\sqrt{n}})^2 + \frac{1}{3}(\frac{z}{i\sqrt{n}})^3} \right)^2$$

$$= \prod_{n \geq 1} \left(1 + \frac{z^2}{n}\right)^2.$$

This can be seen by applying the convergence criterion of Thm.8.2.2 in Greene-Krantz with $a_n = \pm i\sqrt{n}$ and $p_n = 3$ for all n.

Now the function

$$F(z) := f(z)g(z)$$

is an entire function with the required zeros, and no other zeros.