

Uniform convergence on compact sets

1. Show that the sequence $f_n(z) = \frac{1-z^{n+1}}{1-z}$ converges pointwise to $\frac{1}{1-z}$, for $z \in \Delta$. Show that the convergence is locally uniform but not uniform on Δ .

Sol.: For every $z_0 \in \Delta$, the sequence converges pointwise: $f_n(z_0) \rightarrow \frac{1}{1-z_0}$.

The convergence is locally uniform: for all z with $|z| \leq r < 1$

$$\left| \frac{1-z^{n+1}}{1-z} - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{|z|^{n+1}}{|1-|z||} \leq \frac{r^{n+1}}{r-1} \rightarrow 0.$$

The convergence is not uniform on Δ : for n_0 fixed,

$$\lim_{\substack{x \rightarrow 1 \\ x \in \mathbf{R}}} \frac{x^{n_0+1}}{|1-x|} = \infty.$$

Therefore, in order to make $\frac{|z|^{n+1}}{|1-z|}$ small one needs to choose n_0 bigger and bigger as $|z| \rightarrow 1$.

2. Consider the following sequences of functions $\mathbf{C} \rightarrow \mathbf{C}$

$$f_n(z) = \frac{z}{n}, \quad g_n(z) = z^n, \quad h_n(z) = \frac{1}{n^z}.$$

For each sequence:

- (a) Find the pointwise limit;
- (b) Find a set A where the convergence is uniform;
- (c) Find a set U where the convergence is locally uniform;
- (d) Determine whether the convergence is uniform on U or not.

Sol.: (a) The sequence converges pointwise to the function identically zero: in fact for all $z_0 \in \mathbf{C}$, one has $\lim_{n \rightarrow \infty} \frac{z_0}{n} = 0$.

The convergence is locally uniform: for $|z| \leq R$ one has

$$\frac{|z|}{n} \leq \frac{R}{n} \rightarrow 0.$$

The convergence is not uniform on \mathbf{C} : the above estimate cannot be extended uniformly to all \mathbf{C} .

(b) Pointwise, the sequence g_n behaves as follows

$$g_n(z_0) \rightarrow \begin{cases} 0, & |z_0| < 1; \\ \infty, & |z_0| > 1; \\ \text{may have no limit,} & |z_0| = 1. \end{cases}$$

On the unit disk Δ , the convergence is locally uniform, but not uniform : for $|z| \leq r < 1$

$$|z|^n \leq r^n \rightarrow 0.$$

(c) One has $\left|\frac{1}{n^z}\right| = \frac{1}{n^x}$, where $z = x + iy$. Pointwise the sequence h_n behaves as follows

$$h_n(z_0) \rightarrow \begin{cases} 0, & x_0 > 0; \\ 1, & x_0 = 0; \\ +\infty, & x_0 < 0. \end{cases}$$

Hence h_n converges pointwise to 0 on $\{Re(z) > 0\}$. There the convergence is locally uniform but not uniform. On the set $\{Re(z) > r > 0\}$, one has

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^x} \leq \frac{1}{n^r} \rightarrow 0.$$

3. Let $\{p_n\}_{n \in \mathbf{N}}$ be a sequence of polynomials of degree $\leq N$, where N is a fixed positive integer. Show that if $p_n \rightarrow p$ uniformly on compact sets, then the limit function p is a polynomial of degree $\leq N$.

Sol.: A function $p(z)$ is a polynomial of degree $\leq N$ if and only if the k^{th} derivative $p^{(k)}(z)$ is identically 0, for all $k > N$. If the polynomials p_n converge to p uniformly on compact sets, then they converge together with all the derivatives. Let K be a compact set. Then for every $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$

$$|p_n^{(k)}(z) - p^{(k)}(z)| < \epsilon, \quad \forall n > n_0, \quad \forall z \in K.$$

It follows that $p^{(k)}(z) \equiv 0$, for all $k > N$ and $p(z)$ is a polynomial of degree $\leq N$.

4. Let D be a domain and let $f_n: D \rightarrow \mathbf{C}$ be a sequence of holomorphic functions converging to a non-constant function f uniformly on compact sets. Show that if f has m zeros in D , then all but finitely many f_n have at least m zeros in D .

Sol.: Let z_1, \dots, z_l be the zeros of $f \neq 0$. By a reformulation of Hurwitz theorem, for all $j = 1, \dots, l$ there exists n_j such that $f_n(z_j) = 0$, for all $n > n_j$. Then for all $n > m = \max\{n_1, \dots, n_l\}$ one has $f_n(z_j) = 0$. Hence the number of zeros of f_n is $\geq m$, for all $n > m$.

Remark. Let z_0 be a zero of a non-constant holomorphic function f . Then there exists a disk $D(z_0, r)$ such that $f \neq 0$ on $\overline{D(z_0, r)} \setminus \{z_0\}$. Let γ be a contour in $\overline{D(z_0, r)} \setminus \{z_0\}$. If the sequence f_n converges uniformly on compact sets to f , then for n sufficiently large $f_n \neq 0$ on γ and one has

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'_n(\zeta)}{f_n(\zeta)} d\zeta \rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)} d\zeta = p \in \mathbf{N},$$

where p is the multiplicity of z_0 . Therefore also $\frac{1}{2\pi i} \int_{\gamma} \frac{f'_n(\zeta)}{f_n(\zeta)} d\zeta = p$.

5. Let $g: D(0, \rho) \rightarrow \mathbf{C}$ be holomorphic. Let $G: D(0, \rho) \rightarrow \mathbf{C}$ be such that $G' = g$ and $G(0) = 0$. Show that for all $z \in D(0, \rho)$ one has

$$|G(z)| \leq |z| \sup\{|g(w)| : |w| \leq |z|\}.$$

Deduce that $\|G\|_{D(0, \rho)} \leq \rho \|g\|_{D(0, \rho)}$ (suggestion: take $G(z) := \int_{\gamma_z} g(w) dw$, where $\gamma_z(t) = tz$, for $t \in [0, 1]$).

Sol.: As $D(0, r)$ is a convex set, every holomorphic function g on $D(0, r)$ admits a holomorphic primitive (determined up to a constant)

$$G: D(0, r) \rightarrow \mathbf{C}, \quad G'(z) = g(z), \quad G(0) = 0.$$

For example one can take

$$G(z) = \int_{\gamma_z} g(w)dw,$$

where $\gamma_z: [0, 1] \rightarrow D(0, r)$ is the segment $\gamma_z(t) = tz$ joining 0 and z , namely

$$G(z) = \int_0^1 g(tz)zdt.$$

Now from the estimate

$$|G(z)| \leq |z| \sup_{\gamma_z} \{|g(tz)|\} \leq |z| \sup\{|g(w)| : |w| \leq |z|\}, \quad \forall z \in D(0, r),$$

we obtain

$$\|G\|_{D(0, r)} \leq r \|g\|_{D(0, r)}.$$

6. Let f_n be a sequence of holomorphic functions converging locally uniformly to f on $D(0, R)$. Show that $F_n \rightarrow F$ locally uniformly on $D(0, R)$, where $F'_n = f_n$, $F_n(0) = 0$ for all n , and $F' = f$, $F(0) = 0$.

Sol.: From Exercise 5, for all $r_0 < R$ we have the following estimate on $D(0, r_0)$:

$$\|F\|_{D(0, r_0)} \leq r_0 \|f\|_{D(0, r_0)}.$$

In particular

$$\|F_n - F\|_{D(0, r_0)} \leq r_0 \|f_n - f\|_{D(0, r_0)}.$$

Hence $\|f_n - f\|_{D(0, r_0)} \rightarrow 0$ implies $\|F_n - F\|_{D(0, r_0)} \rightarrow 0$, as claimed.

7. Discuss the convergence and uniform convergence of the sequence $f_n(z) = nz^n$, for $n \in \mathbf{N}$.

Sol.: Pointwise, the sequence behaves as follows

$$n|z_0|^n \rightarrow \infty, \quad \forall z_0 \quad |z_0| \geq 1, \quad n|z_0|^n \rightarrow 0, \quad \forall z_0 \quad |z_0| < 1.$$

On the unit disk Δ , the convergence is locally uniform, but not uniform. In fact, for $|z| \leq r < 1$.

$$n|z|^n \leq nr^n = \frac{n}{\frac{1}{r^n}} \rightarrow 0.$$

8. Prove that the sequence $f_n(z) = \frac{1}{1+nz}$ is uniformly convergent to the function identically 0, for all z such that $|z| > 2$. Can the region of uniform convergence be extended?

Sol.: For every fixed $n \in \mathbf{N}$, the function f_n is a meromorphic function with a simple pole at $z = -\frac{1}{n}$. Hence all the poles of the sequence are contained in the closed disk $\overline{D(-\frac{1}{2}, \frac{1}{2})}$. Then all the functions f_n are holomorphic on the set $\{z \in \mathbf{C} : |z| > 2\}$ and on that set the sequence converges pointwise to the function identically 0. Fix $R > \frac{1}{2}$. The estimate

$$|nz + 1| \geq |nR - 1| \rightarrow \infty, \quad \text{for } n \rightarrow \infty,$$

which is independent of z , shows that the convergence is locally uniform, and therefore uniform on compact sets.

Actually, the sequence converges locally uniformly on \mathbf{C}^* . Given an arbitrary fixed compact set K , it is contained in some set of the form $\{z \in \mathbf{C} : |z| > r\}$, for some $r > 0$. After removing finitely many functions (depending on r), we are left with a family of holomorphic functions on $\{z \in \mathbf{C} : |z| > r\}$ and estimates similar to the above ones show the local uniform convergence.

The point $z = 0$ cannot be included in the convergence set (not even pointwise) since all functions have value 1 at $z = 0$.

9. Let \mathcal{F} be the family of all analytic functions on the open unit disc Δ whose coefficients in the Taylor expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots,$$

satisfy $|a_n| \leq n$, for each n . Show that \mathcal{F} is relatively compact (with respect to the topology of locally uniform convergence).

Sol.: We need to show that every family of functions f_n in \mathcal{F} admits a subsequence f_{n_j} converging to an element $f_0 \in \mathcal{O}(\Delta)$. By Montel's theorem, it is sufficient to show that \mathcal{F} is equibounded on compact sets (equivalently, locally equibounded). Fix $D(0, r) \subset \Delta$, for $0 < r < 1$. Let $f \in \mathcal{F}$. Then

$$|f(z)| \leq 1 + \sum_{n \geq 1} |a_n| r^n \leq 1 + \sum_{n \geq 1} n r^n, \quad \forall z, |z| \leq r.$$

Since the series $\sum_{n \geq 1} n r^n$ converges, then

$$|f(z)| \leq S, \quad \forall z, |z| \leq r.$$

This shows that the family \mathcal{F} is locally equibounded and therefore equibounded on compact sets.

10. Let f_n be a sequence of analytic functions on Δ , uniformly bounded. Assume that for each $z \in \Delta$ the sequence $f_n(z)$ converges. Show that f_n converges uniformly on compact subsets of Δ .

Sol.: We are going to show that the sequence f_n is uniformly Cauchy on discs $D(z_0, r)$ in Δ . This is sufficient since any compact set in Δ can be covered by finitely many disks of a fixed radius.

Let $z \in D(z_0, r)$. Write

$$\begin{aligned} |f_n(z) - f_m(z)| &= |f_n(z) - f_n(z_0) + f_n(z_0) - f_m(z_0) - (f_m(z) - f_m(z_0))| \\ &\leq |f_n(z) - f_n(z_0)| + |f_n(z_0) - f_m(z_0)| + |f_m(z) - f_m(z_0)|. \end{aligned} \quad (*)$$

Since f_n is uniformly bounded, then it is uniformly continuous (cf. proof of Montel's theorem). Namely if $z, z_0 \in \Delta$, then

$$|f(z) - f(z_0)| \leq M|z - z_0|, \quad \text{for some } M \in \mathbf{R}. \quad (**)$$

By (**), we may choose r such that the first two summands of (*) are $< \epsilon/3$, for all n ; since the sequence is pointwise convergent, we may choose N such that for all $n, m > N$, the third summand is $< \epsilon/3$. As a result, for r sufficiently small and n sufficiently large (depending on ϵ)

$$|f_n(z) - f_m(z)| \leq M|z - z_0| + M|z - z_0| + |f_n(z_0) - f_m(z_0)| < \epsilon.$$

11. Consider the family $\mathcal{F} = \{f_n(z) = \frac{z}{n}\}_{n \in \mathbf{N}}$, defined on \mathbf{C} . Verify that \mathcal{F} is not equibounded on \mathbf{C} , but it is equibounded on compact sets. Indeed it converges uniformly on compact sets to $f \equiv 0$.

Sol.: For all $|z| \leq R$ and for all $n \geq 1$ one has

$$\left| \frac{z}{n} \right| \leq R.$$

Since every compact set K is contained in some disk $D(0, R)$, for some $R > 0$, then the family \mathcal{F} is equibounded on compact sets. But it is not equibounded on all \mathbf{C} .

By Montel's theorem, every sequence in \mathcal{F} admits a subsequence converging to a function $f: \mathbf{C} \rightarrow \mathbf{C}$. In this case \mathcal{F} itself converges uniformly on compact sets to $f \equiv 0$.

Some examples for comparing the real and the complex cases.

- a. (*Weierstrass' convergence theorem does not hold in the real case*) Consider the sequence of functions $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, defined on $[-1, 1]$. Verify that the functions f_n are smooth, and converge uniformly to a non-smooth function f .
- b. (*Hurwitz's theorem does not hold in the real case*) Consider the sequence $f_n: \mathbf{R} \rightarrow \mathbf{R}$, defined by $f_n(x) = x^2 + \frac{1}{n}$. Verify that the functions f_n are never 0 on \mathbf{R} . Nevertheless they converge uniformly on compact sets to $f(x) = x^2$, which takes the value $f(0) = 0$.
- c. (*Montel's theorem does not hold in the real case*) Consider the family $\mathcal{F} = \{f_n(x) = \sin(nx)\}_n$, defined on $[0, 2\pi]$. Verify that \mathcal{F} is equibounded. Nevertheless it does not admit any converging subsequence (not even pointwise).