Laurent series and isolated singularities, Riemann extension theorem, Casorati-Weierstrass theorem, automorphisms of the plane, residue theorem, Rouché theorem.

1. *Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions with $g(z) \neq 0$, for every $z \in \mathbb{C}$. Assume that $|f(z)| \leq |g(z)|$, for every $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ (cf. Exercise 17 in sheet 0). Show that the assumption "$g(z) \neq 0$, for every $z \in \mathbb{C}$" can be replaced with "g not identically zero".*

Sol.: Under the assumption that g never vanishes, f/g is a bounded holomorphic function. Hence it is constant by Liouville's theorem, i.e. $f/g = c$.

Without such assumption, the zeros of g are isolated singularities of f/g. However f/g is bounded near such singularities. By the Riemann extension theorem, they are removable singularities and the statement follows from Liouville's theorem.

2. *(a) Let f be a bounded holomorphic function defined on $\mathbb{C} \setminus \{0, i, 1+i\}$ or on $C \setminus \mathbb{Z}$. Show that f is constant.*
   (b) Is it true that f is constant when it is bounded on $\mathbb{C} \setminus (\{1/n : n \in \mathbb{N}\} \cup \{0\})$?

Sol.: (a) The boundedness assumption implies that all singularities of f are removable. In other words, f is holomorphic and bounded on all $\mathbb{C}$. Therefore it is constant.

(b) By the boundedness assumption, all isolated singularities $\{1/n\}_{n\in\mathbb{N}}$ are removable. Hence f is holomorphic and bounded at least on $\mathbb{C} \setminus \{0\}$. On the other hand, if 0 were singular, then f could not be bounded in any neighbourhood of 0. Conclusion: f is holomorphic and bounded on all $\mathbb{C}$, and therefore constant.

3. *Show that the functions*

$$\tfrac{\sin z}{z}, \qquad \tfrac{e^z-1}{z}, \qquad \tfrac{\cosh z - 1}{z}$$

   are entire, i.e. they extend holomorphically to $\mathbb{C}$.

Sol.: The above functions are holomorphic on all $\mathbb{C}$, except possibly for $z = 0$. By expanding the numerators around $z = 0$ we find

$$\frac{\sin z}{z} = \frac{z - z^3/3! + z^5/5! - \dots}{z} = 1 - z^2/3! + z^4/5! - \dots$$

$$\frac{e^z - 1}{z} = \frac{1 + z + z^2/2! + z^3/3! + \dots - 1}{z} = 1 + z/2! + z^2/3! + \dots$$

$$\frac{\cosh z - 1}{z} = \frac{1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \dots - 1}{z} = \frac{1}{2}z + \frac{1}{24}z^3 + \dots,$$

which show that $z = 0$ is indeed a removable singularity, i.e. the three functions are entire.

4. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.*
   (a) Show that if f has a zero of order n in z_0, then $1/f$ has a pole of order n in z_0.
   (b) Let z_0 be a singularity (removable, polar, essential) of f. Determine the corresponding type of singularity of $1/f$.

Sol.: (a) Write $f(z)=(z-z_0)^n u(z)$, where $u(z)=a_n+a_{n+1}(z-z_0)+\dots$ is a holomorphic function with $u(z_0)\neq 0$. Then

$$1/f(z)=\frac{1}{(z-z_0)^n}\frac{1}{u(z)}.$$

Since $1/u(z)$ is holomorphic around z_0, then z_0 is a pole of $1/f$ of order n.

(b) Let z_0 be a removable singularity for f and let $f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\dots$ be the Taylor expansion of f around z_0. If $f(z_0)\neq 0$, then $1/f$ is holomorphic around z_0; if z_0 is a zero of f of order n, then z_0 is a pole of order n of $1/f$.

Let z_0 be a pole of f or order m and let

$$f(z)=\frac{a_{-m}}{(z-z_0)^m}+\frac{a_{-m+1}}{(z-z_0)^{m-1}}\dots=\frac{1}{(z-z_0)^m}(a_{-m}+a_{-m+1}(z-z_0)+\dots)$$

be the Laurent expansion of f around z_0, where $g(z)=a_{-m}+a_{-m+1}(z-z_0)+\dots$ is a holomorphic function with $g(z_0)\neq 0$. Now it is clear that z_0 is a zero of $1/f=\frac{(z-z_0)^m}{g(z)}$ of order n.

(c) Let z_0 be an essential singularity for f. Then the image of any disc $D(z_0,r)\setminus\{z_0\}$ is dense in $\mathbb{C}$ and the same is true for $\frac{1}{f}$. Hence z_0 is an essential singularity for $1/f$ as well.

5. *Let z_0 be a pole of f. Recall that for every $M>>0$ there exists $\varepsilon>0$ such that $f(D^*(z_0,\epsilon))\subset\{|z|>M\}$.*
   Show that, given $\varepsilon>0$, there exists $M>>0$ such that $\{|z|>M\}\subset f(D^*(z_0,\epsilon))$.
   (Justify your answer and mention all the results you used.)

Sol.: If z_0 is a pole, then $\lim_{z\to z_0}|f(z)|=+\infty$. Equivalently, for every $M>>0$ there exists $\varepsilon>0$ such that $f(D^*(z_0,\epsilon))\subset\{|z|>M\}$.

View the map f valued in $\mathbb{P}^1(\mathbb{C})$, with $f(z_0)=\infty$. One easily cheks that f is a non constant holomorphic map to $\mathbb{P}^1(\mathbb{C})$, viewed as a Riemann surface. In particular f is open and its image contains a neighbourhood of ∞. Therefore, given $\varepsilon>0$, there exists $M>>0$ such that $\{|z|>M\}\subset f(D^*(z_0,\epsilon))$.

6. *Determine and classify all the singularities of the following functions:*

$$\tan z,\quad \tfrac{1}{z^3}\textstyle\sum_{n=2}^{\infty}2^n z^n,\quad ze^{1/z}e^{-1/z^2},\quad \frac{\sin\frac{1}{z}}{z^4},\quad \frac{\sin z}{z^4},\quad \frac{1}{z^3}-\cos z.$$

Sol.: (a) The function $f(z)=\tan z=\sin z/\cos z$ is defined and holomorphic on $\mathbb{C}\setminus\{\pi/2+k\pi\},\ k\in\mathbb{Z}\}$, where $\cos z\neq 0$, and it is periodic of period π. So it is sufficient to determine the type singularity at $\pi/2$. For this note that $\sin z$ is non-zero close to $\pi/2$ while $\cos z$ has a zero of order one in $\pi/2$. It follows that all points $\{\pi/2+k\pi,\ k\in\mathbb{Z}\}$ are simple poles for $\tan z$ (cf. ex. 4).

(b) The function

$$\frac{1}{z^3}\sum_{n=2}^{\infty}2^n z^n=\sum_{n=2}^{\infty}2^n z^{n-3}=2^2\frac{1}{z}+2^3+2^4z+\dots$$

has a pole of order 1 at $z=0$.

(c) The function

$$ze^{1/z}e^{-1/z^2}=ze^{1/z-1/z^2}=z(1+(\frac{1}{z}-\frac{1}{z^2})+\frac{1}{2}(\frac{1}{z}-\frac{1}{z^2})^2+\dots)$$

has an essential singularity at $z = 0$.

(d) The function

$$\frac{\sin(1/z)}{z^4} = \frac{1}{z^4}(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \dots)$$

has an essential singularity at $z = 0$.

(e) The function

$$\frac{\sin z}{z^4} = \frac{1}{z^4}(z - \frac{1}{3!}z^3 + \dots)$$

has a pole of order 3 at $z = 0$. Alternatively, one can observe that

$$\lim_{z \to 0} z^3 \frac{\sin z}{z^4} = \lim_{z \to 0} \frac{\sin z}{z} = 1.$$

(f) Since $\cos z$ is holomorphic on $\mathbb{C}$, the function

$$\frac{1}{z^3} - \cos z$$

has a pole of order 3 at $z = 0$.

7. *Let* $g(z) := e^{1/z} - e^{2/z}$. *Determine* $g(\mathbb{C} \setminus \{0\})$.

Sol.: The Laurent series expansion of g around $z = 0$ is given by

$$g(z) = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \dots - (1 + \frac{2}{z} + \frac{1}{2!}\frac{2^2}{z^2} + \frac{1}{3!}\frac{2^3}{z^3} + \dots$$
$$= -\frac{1}{z} - \frac{1}{2!}\frac{2^2-1}{z^2} - \frac{1}{3!}\frac{2^3-1}{z^3} + \dots$$

One sees that $z = 0$ is an essential singularity for g. Hence $g(\mathbb{C} \setminus \{0\})$ is dense in $\mathbb{C}$, by Casorati-Weierstrass theorem. Also, the equation $w = e^{1/z}(1 - e^{1/z})$ can be solved for every $w \in \mathbb{C}$. In order to see this, set $\eta := e^{1/z}$. Then such equation reads as $\eta^2 - \eta + w = 0$. For $w \neq 0$ fixed, one obtains two different roots in the punctured complex plane $\mathbb{C}^*$, which is the image of the map $z \to e^{\frac{1}{z}}$. For $w = 0$, one root is $\eta = 1$ and, indeed, for $\frac{1}{z} = 2\pi i k$, with $k \neq 0$, one obtains $e^{1/z} - e^{2/z} = 0$. Therefore $g(\mathbb{C} \setminus \{0\}) = \mathbb{C}$.

8. *Let* $f(z) = \frac{z^2}{z^2+1}$. *Determine the Laurent serie of* f *around* $z_0 = i$.

Sol.: The function $f(z) = \frac{z^2}{(z+i)(z-i)}$ is holomorphic on $\mathbb{C}$, except for $\pm i$ where it has simple poles. The Laurent serie of f around $z_0 = i$ is given by

$$f(z) = \frac{1}{(z-i)} \sum_{n \geq 0} a_n (z-i)^n,$$

where $\sum_{n \geq 0} a_n (z-i)^n$ is the Taylor series expansion of $g(z) = \frac{z^2}{(z+i)}$, which is holomorphic around $z_0 = i$. The coefficients of such series are

$$a_0 = g(i) = i/2, \quad a_1 = g'(i) = 3/4, \dots, \qquad a_n = \frac{1}{n!} g^{(n)}(i).$$

9. *Let f be a holomorphic function on $D^*(0,r)$, for $0 < r \leq +\infty$. Assume that $z_0 = 0$ is a pole of f of order m. Then the Laurent serie of f around z_0 is given by*

$$f(z) = \textstyle\sum_{n \geq -m} a_n z^n, \qquad a_n = \frac{1}{(n+m)!} g^{(n+m)}(0), \quad \text{where } g(z) = z^m f(z).$$

Sol.: The Laurent series expansion of f around $z = 0$ is given by

$$f(z) = \frac{a_{-m}}{z^m} + \frac{a_{-(m-1)}}{z^{m-1}} + \ldots = \frac{1}{z^m}(a_{-m} + a_{-(m-1)}z + \ldots),$$

and $g(z) = z^m f(z) = b_0 + b_1 z + \ldots$ is a holomorphic function with $g(z) \neq 0$. The coefficients of the series expansion of g are

$$b_k = a_{-m+k} = \frac{1}{k!} g^{(k)}(0).$$

If $n = -m + k$, then

$$a_n = \frac{1}{(n+m)!} g^{(n+m)}(0).$$

10. *Let $f(z) = \frac{e^z}{(z-i)^3(z+2)^2}$.*
    (i) *Show that $z_1 = i$ is a pole of f of order 3.*
    (ii) *Determine all the coefficients of the principal part of the Laurent series of f around z_1.*
    (iii) *Show that $z_2 = -2$ is a pole of f of order 2.*
    (iv) *Determine all the coefficients of the principal part of the Laurent series of f around z_2.*

Sol.: (i) The limit $\lim_{z \to i}(z-i)^3 \cdot f(z) = \frac{e^i}{(i+2)^2}$ is finite. Hence $z_1 = i$ is a pole of f of order 3.

(ii) The Laurent series expansion of f around $z_1 = i$ is given by

$$\frac{1}{(z-i)^3} \sum_{n \geq 0} a_n (z-i)^n,$$

where $\sum_{n \geq 0} a_n (z-i)^n$ is the Taylor series expansion of the holomorphic function $g(z) = \frac{e^z}{(z+2)^2}$ around z_1. The principal part of the Laurent series of f around z_1 is

$$a_{-3} \frac{1}{(z-i)^3} + a_{-2} \frac{1}{(z-i)^2} + a_{-1} \frac{1}{(z-i)}.$$

The coefficients can be computed as in Exercise 9

$$a_{-3} = g(i) = \frac{e^i}{(i+2)^2}, \qquad a_{-2} = g'(i) = \frac{e^i(1+i)}{(i+2)^2} \qquad a_{-1} = \frac{1}{2!} g''(i) = \ldots.$$

(iii) The limit $\lim_{z \to i}(z+2)^2 \cdot f(z) = \frac{e^{-2}}{(-2-i)^3}$ is finite. Hence $z_2 = -2$ is a pole of f of order 2.

(iv) The Laurent series expansion of f around $z_2 = -2$ is given by

$$\frac{1}{(z+2)^2} \sum_{n \geq 0} a_n (z+2)^n,$$

where $\sum_{n\geq 0} a_n(z+2)^n$ is the Taylor series expansion of the holomorphic function $g(z) = \frac{e^z}{(z+2)^3}$ around z_2. The principal part of the Laurent series of f around z_2 is

$$a_{-2}\frac{1}{(z+2)^2} + a_{-1}\frac{1}{(z+2)}.$$

The coefficients can be computed as in Exercise 9

$$a_{-2} = g(-2), \qquad a_{-1} = g'(-2).$$

11. *Let $\gamma_2(\theta) = e^{i\theta}$, where $\theta \in [0, 2\pi]$, and $\gamma_1(\theta) = 3e^{i\theta}$, where $\theta \in [0, 2\pi]$. Compute*

$$(a) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2+5z}{(z-2)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2+5z}{(z-2)}dz;$$

$$(b) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2-2}{z}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2-2}{z}dz;$$

$$(c) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^3-3z-6}{z(z+2)(z+4)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^3-3z-6}{z(z+2)(z+4)}dz.$$

Sol.: The curves γ_1 and γ_2 are circles centered in the origin, oriented counterclockwise, of radius 3 and radius 1, respectively. (a) The function $f(z) = \frac{z^2+5z}{(z-2)}$ is holomorphic on $\mathbb{C} \setminus \{2\}$ and has a pole of order 1 in $z = 2$ with residue $Res_f(2) = \lim_{z\to 2}(z-2)f(z) = 14$. The pole is inside γ_1 and outside γ_2.
Hence the integral in (a) is equal to $2\pi i Res_f(2) = 28\pi i$.
(b) The function $f(z) = \frac{z^2-2}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and has a pole of order 1 in $z = 0$ with residue $Res_f(0) = \lim_{z\to 0}(z)f(z) = -2$. The pole is inside γ_1 and inside γ_2.
Hence the integral in (b) is equal to 0.
(c) The function $f(z) = \frac{z^3-3z-6}{z(z+2)(z+4)}$ has simple poles in $z = 0$, inside both circles, $z = -2$ inside γ_1 and outside Γ_2, and $z = -4$, outside both circles.
Hence the integral in (c) is given by $2\pi i(Res_f(0) + Res_f(-2) - Res_f(0)) = 2\pi i Res_f(-2) = 4\pi i$.

12. *Compute the residues of the following functions at the singular points:*

$$e^{3/z^2}, \qquad \frac{z^3}{z-1}, \qquad \frac{z^3}{(z-1)^2}, \qquad \frac{z^3}{1-z^4}, \qquad \frac{z^5}{(z^2-1)^2}, \qquad \frac{\cos z}{1+z+z^2}, \qquad \frac{1}{\sin z}.$$

Sol.:

$$f(z) = e^{3/z^2} = 1 + \frac{3}{z^2} + \frac{1}{2}(\frac{3}{z^2})^2 + \ldots, \qquad Res_f(0) = 0$$

$$f(z) = \frac{z^3}{z-1}, \qquad Res_f(1) = \lim_{z\to 1}(z-1)f(z) = 1;$$

$$f(z) = \frac{z^3}{(z-1)^2} = \frac{1}{(z-1)^2} + \frac{2}{(z-1)} + \ldots, \qquad Res_f(1) = 2;$$

$$f(z) = \frac{z^3}{1-z^4} = \frac{z^3}{(z-1)(z+1)(z-i)(z+i)} \qquad Res_f(1) = \lim_{z\to 1}(z-1)f(z) = \frac{1}{4}, \quad etc\ldots$$

$$f(z) = \frac{z^5}{(z^2-1)^2} = \frac{z^5}{(z+1)^2(z-1)^2} = \frac{1}{(z+1)^2}\left(-\frac{1}{4} + (z+1) + \ldots\right) \quad Res_f(-1) = 1;$$

$$f(z) = \frac{z^5}{(z^2-1)^2} = \frac{1}{(z-1)^2}\left(\frac{1}{4} + (z-1) + \ldots\right) \quad Res_f(1) = 1;$$

The points $\alpha = -1/2 + i\sqrt{3}/2$ and $\beta = -1/2 - i\sqrt{3}/2$ are simple poles for f

$$f(z) = \frac{\cos z}{1+z+z^2} = \frac{\cos z}{(z-\alpha)(z-\beta)}.$$

Then

$$Res_f(\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \frac{\cos\alpha}{\alpha-\beta}, \qquad Res_f(\beta) = \lim_{z\to\beta}(z-\beta)f(z) = \frac{\cos\beta}{\beta-\alpha}.$$

$$f(z) = \frac{1}{\sin z} \qquad Res_f(0) = Res_f(k\pi) = 1, \quad \forall k \in \mathbb{Z}.$$

13. *Show that the function defined by* $\sum_{n=1}^{\infty} \frac{1}{n! z^n}$ *is holomorphic on* $\mathbb{C}\setminus\{0\}$. *Compute its integral on* $\gamma(\theta) = e^{i\theta}$, *for* $\theta \in [0, 2\pi]$.

Sol.: The function defined by $\sum_{n=1}^{\infty} \frac{1}{n! z^n}$ is just $e^{1/z}$, which is holomorphic on $\mathbb{C}\setminus\{0\}$. It has an essential singularity at $z = 0$ with residue 1. Hence the integral on f on γ is equal to $2\pi i$.

14. *Compute*

$$\int_\gamma \frac{e^z}{z^3}dz, \qquad \int_\gamma \frac{e^{1/z}}{z^3}dz,$$

when $\gamma(\theta) = e^{i\theta}$, *for* $\theta \in [0, 6\pi]$ *and when* $\gamma(\theta) = e^{i\theta}$, *for* $\theta \in [0, 2\pi]$.

Sol.: The curve $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 2\pi]$, is the unit circle, oriented counterclockwise, covered 1 time. Then

$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \ldots, \qquad \int_\gamma f(z)dz = 2\pi i Res_f(0) = 2\pi i\frac{1}{2} = \pi i,$$

$$f(z) = \frac{e^{1/z}}{z^3} = \frac{1}{z^3} + \frac{1}{z^4} + \ldots, \qquad \int_\gamma f(z)dz = 2\pi i Res_f(0) = 0.$$

The curve $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 6\pi]$, is the unit circle, oriented counterclockwise, covered 3 times. Then

$$f(z) = \frac{e^z}{z^3}, \qquad \int_\gamma f(z)dz = 3\cdot 2\pi i Res_f(0) = 3\pi i,$$

$$f(z) = \frac{e^{1/z}}{z^3}, \qquad \int_\gamma f(z)dz = 3\cdot 2\pi i Res_f(0) = 0.$$

15. *Use the "sector method" to compute*

$$\int_0^{+\infty} \frac{1}{1+x^4} dx .$$

Sol.: See Sarason, Example 3, p.126.

16. *Let c be a complex number with $|c| > e$. Show that the equation $e^z = cz^n$ has n solutions in the unit disk.*

Sol.: We apply Rouché's theorem to $g(z) = -cz^n$ and $h(z) = e^z$. For $|z| = 1$, one has

$$|g(z)| = |c| > e > e^{\mathrm{Re} z} = |h(z)|.$$

Then the number of zeros of $g + h$ is equal to the number of zeros of g in the unit disk. Therefore the equation $e^z = cz^n$ has n solutions in the unit disk.

17. *Show that $z^5 + 15z + 1 = 0$ has 4 solutions in the annulus $L_{3/2,2}(0) = \{3/2 < |z| < 2\}$.*

Sol.: We first apply Rouché's theorem to $g(z) = z^5$ and $h(z) = 15z + 1$. For $|z| = 2$, one has

$$|h(z)| = |15z + 1| \leq |15z| + 1 < 31 < 2^5 = |g(z)|.$$

Then the number of zeros of $g + h$ is equal to the number of zeros of g in the disk of radius 2, namely $g + h$ has 5 zeros.

Next we show that $z^5 + 15z + 1 = 0$ has only one solution inside the disk of radius $3/2$. We apply Rouché's theorem to $g(z) = 15z$ and $h(z) = z^5 + 1$. For $|z| = 3/2$, one has

$$|h(z)| = |z^5 + 1| \leq |z|^5 + 1 = (\tfrac{3}{2})^5 + 1 < 15(\tfrac{3}{2}) < |g(z)|.$$

Therefore $g + h$ has a single zero in the disk of radius $3/2$.

A remark by Ahmed Yacine: since the polynomial has real coefficients, every non-real root comes together with its conjugate. Moreover, the disks are invariant with respect to conjugation. It follows that the unique root inside the disk of radius $3/2$ is real.