**Liouville's theorem, identity principle, maximum modulus principle, harmonic functions, Schwarz's lemma, automorphisms of the unit disk.**

1. *Let $f\colon\mathbb{C}\longrightarrow\mathbb{C}$ be a holomorphic doubly periodic function (it means that $f(z+\omega_1)=f(z+\omega_2)=f(z)$ for all $z\in\mathbb{C}$, where $\omega_1,\omega_2\in\mathbb{C}$ are $\mathbb{R}$-linearly independent vectors). Then $f$ is constant.*

*Sol.*: A doubly periodic function is necessarily bounded, as it takes all its values on the compact set $P=\{z=a\omega_1+b\omega_2\ :\ a,b\in[0,1]\}$. Hence it is constant by Liouville theorem.

2. *Let $U\subset\mathbb{C}$ be an open neighbourhood of $0$. Show that there are no (non-constant) holomorphic functions $f\colon U\to\mathbb{C}$, such that:*
   *(a)* $f(\frac{1}{n})=(-1)^n\frac{1}{n^2}$,
   *(b)* $f(\frac{1}{n})=\frac{1}{2^n}$,
   *(c)* $|f^{(n)}(0)|>n!\,n^n$.

*Sol.*: (a) $f(\frac{1}{n})$ coincides with $g(z)=z^2$, for $n$ even, while it coincides with $h(z)=-z^2$, for $n$ odd. Since the sets

$$\{\frac{1}{2k}\}\cup\{0\}\qquad\text{and}\qquad\{\frac{1}{2k+1}\}\cup\{0\},\qquad k\in\mathbb{N}$$

are uniqueness sets for $f$, there is no holomorphic function satisfying $f(\frac{1}{n})=(-1)^n\frac{1}{n^2}$.
(b) We are going to show that $f$ is identically zero, by showing that all the derivatives of $f$ at $z=0$ are zero. We have

$$f(0)=\lim_{n\to+\infty}\frac{1}{2^n}=0\qquad\text{and}\quad f'(0)=\lim_{n\to+\infty}\frac{\frac{1}{2^n}-0}{1/n}=0.$$

Now we proceed by induction. We assume that $f^{(h)}(0)=0$, for all $h\le k$, and prove that $f^{(k+1)}(0)$. By our assumption

$$f(z)=a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\ldots=z^{k+1}(a_{k+1}+a_{k+2}z+\ldots).$$

Define $g(z)=\frac{f(z)}{z^{k+1}}$. Then $f^{(k+1)}(0)$ if and only if $g(0)=0$. Indeed

$$g(0)=\lim_{n\to+\infty}\frac{f(1/n)}{\frac{1}{n^{k+1}}}=\lim_{n\to+\infty}\frac{n^{k+1}}{2^n}=0.$$

(c) Let $\sum_{n\ge0}a_nz^n$ be the series expansion of $f$ around $z=0$. Then $a_n=\frac{1}{n!}f^{(n)}(0)$ satisfies $|a_n|>n^n$. The radius of convergence $R$ of such a series is given by

$$\frac{1}{R}=\limsup_{n\to+\infty}|a_n|^{1/n}=\limsup_{n\to+\infty}n=+\infty.$$

It follows that $R=0$ and there is no holomorphic function as in (c).

3. *Let $D$ be a domain in $\mathbb{C}$ and let $f\colon D\to\mathbb{C}$ be a holomorphic function, not identically zero. Prove that the set of zeros of $f$ in $D$ is at most countable (use: $D$ is a countable union of compact sets).*

*Sol.*: The zeros of a holomorphic, not identically vanishing function are discrete. Hence finite in a compact set. As $D$ is a countable union of compact sets, the set of zeros of $f$ in $D$ is at most countable (a countable union of finite sets is countable).

4. *Let* $f: U \to \mathbb{C}$ *be a holomorphic function on an open neighbourhood* $U$ *of the closed unit disc* $\overline{\Delta}$. *Assume that* $f$ *is not identically zero.*
   (a) *Show that* $f$ *has at most finitely many zeros in* $\Delta$.
   (b) *Determine the zeros of* $f(z) = \sin(\frac{1}{1-z})$ *on the disc* $\Delta$ *and compare the result with (a).*

*Sol.*: (a) Since $\overline{\Delta}$ is compact, $f$ has at most finitely many zeros in $\overline{\Delta}$ and therefore in $\Delta$ (we used that the zero set of a holomorphic, not identically vanishing function is closed and consists of isolated points. Hence its intersection with a compact set inside the domain of definition is finite).
(b) $f(z) = \sin(\frac{1}{1-z}) = 0$ if and only if $z = 1 - \frac{1}{k\pi}$, with $k \in \mathbb{Z}$. The zeros in $\Delta$ are the ones with $k \in \mathbb{Z}_{>0}$. They are infinitely many and they accumulate in $z = 1$, which is a singularity of $f$.
*Note*: in this case $f$ is not holomorphic on an open neighbourhood of the closed unit disc $\overline{\Delta}$. Hence the arguments in (a) do not apply.

5. *Let* $S = \{z \in \mathbb{C} \ : \ 0 < \mathrm{Re} z < \frac{\pi}{2}\}$. *Determine whether there exists a holomorphic function* $f: S \to \mathbb{C}$ *such that*
   (a) $\mathrm{Re} f(z) = x^2y + y^2x + \sin x \sinh y$;
   (b) $\mathrm{Re} f(z) = y^3 - 3x^2y + \cos x \cosh y + \sin x \sinh y$.
   *Justify your answer: either exhibit one such function or explain why it cannot exist.*

*Sol.*: On the convex set $S$ a function is the real part of a holomorphic function if and only if it is harmonic.
The function in (a) is not harmonic: it is the sum of the harmonic function $\sin x \sinh y = Im(\cos z)$ and the function $f(x, y) = x^2y + y^2x$, whose Laplacian is given by $\Delta f = 2(x + y) \not\equiv 0$.
The function in (b) is harmonic: it is the sum of the harmonic functions $Re(\cos z) = \cos x \cosh y$, $Im(\cos z) = \sin x \sinh y = -Re(i \cos z)$ and the function $u(x, y) = y^3 - 3x^2y$ whose Laplacian is identically zero. Hence it is the real part of the holomorphic function

$$f(z) = u(x, y) + iv(x, y) + \cos z - i \cos z,$$

where $v(x, y)$ is a harmonic conjugate of $u(x, y)$. We determine $v$ by the Cauchy-Riemann equations:

$$v_y = u_x = -6xy \quad \Rightarrow \quad v(x, y) = -3xy^2 + \phi(x);$$

$$v_x = -3y^2 + \phi'(x) = -u_y = -(3y^2 - 3x^2) \quad \Leftrightarrow \quad \phi'(x) = 3x^2 \quad \Leftrightarrow \quad \phi(x) = x^3 + c, \ c \in \mathbb{R}.$$

Conclusion: $v(x, y) = -3xy^2 + x^3 + c$ and

$$f = y^3 - 3x^2y + i(-3xy^2 + x^3) + \cos z - i \cos z + ic, \quad c \in \mathbb{R}.$$

6. (*Liouville's theorem for harmonic functions*). *Let* $u: \mathbb{C} \to \mathbb{R}$ *be harmonic and bounded either from above or from below.*
   (a) *Show that* $u$ *is constant.*

(b) *Verify that the real and the imaginary parts of the following holomorphic functions are not bounded:*

$$e^z, \qquad \sin z, \qquad \cos z, \qquad z^2.$$

*Sol.*: (a) The harmonic function $u$ is the real part of a holomorphic function

$$f\colon \mathbb{C} \to \mathbb{C}, \quad f(z) = u(z) + iv(z),$$

where $v$ is a harmonic conjugate of $u$. Suppose $u$ is bounded by above. Then so is the absolute value of the holomorphic function $e^{f(z)} = e^{u(z)}e^{iv(z)}$. By Liouville's theorem, $e^{f(z)} \equiv c$ is constant. Then also $f(z) = \log(c)$ is constant (on $\mathbb{C}$ it is possible to choose a global determination of the logarithm, and we did so).

Suppose now that $u$ is bounded by below. Then we can apply the above argument to the harmonic function $-u$.

7. *Set $D = D(z_0, r)$. Let $f\colon D \to \mathbb{C}$ be a holomorphic function. Show that*

$$f(z_0) = \frac{1}{Area(D)} \int_D f(z)dxdy\,.$$

*Sol.*: We compute the integral in polar coordinates. By the substitutions $x = \rho\cos t$, $y = \rho\sin t$, $dxdy = \rho d\rho dt$, the integral $\int_D f(z)dxdy$ becomes

$$\int_0^r \int_0^{2\pi} f(z_0 + \rho e^{it})\rho d\rho dt = \int_0^r \rho \left( \int_0^{2\pi} f(z_0 + \rho e^{it})dt \right) d\rho.$$

By the mean value property, the above integral equals

$$= \int_0^r \rho 2\pi f(z_0)d\rho = 2\pi f(z_0)(\rho^2/2|_0^r) = f(z_0)\pi r^2,$$

from which we deduce

$$f(z_0) = \frac{1}{\pi r^2} \int\int f(x,y)dxdy = \frac{1}{area(D)} \int\int f(x,y)dxdy,$$

as claimed.

8. *Let $D$ be a domain in $\mathbb{C}$ and let $f\colon D \to \mathbb{C}$ be a nonconstant holomorphic function. Show that the local minima of $|f|$ coincide with the zeros of $f$.*

*Sol.*: Let $z_0 \in D$ be a point of local minimum for $|f(z)|$: it means that there is an open neighbourhood $U$ of $z_0$ in $D$ with the property that $|f(z_0)| \leq |f(z)|$, for all $z \in U$. Suppose that $f(z_0) \neq 0$. Then $g(z) = 1/f(z)$ is a holomorphic function of $U$ and $z_0$ is a local maximum for $|g(z)|$. Then $g$ is constant on $U$ and so is $f$. As $D$ is connected, by the identity principle $f$ is constant on $D$. Contradiction.

9. *Let $f: U \to \mathbb{C}$ be a nonconstant holomorphic function defined on a neighbourhood of the unit disc $\Delta$. Show that if $|f|$ is constant on the boundary of $\Delta$, then $f$ admits at least one zero in $\Delta$.*

*Sol.*: The closure of the disc $\overline{\Delta}$ is a compact set. Hence $|f|$ has maximum and minimum on $\overline{\Delta}$. They are distinct because $f$ is non constant. Since the maximum of $|f|$ is attained on the boundary of $\Delta$, then the minimum is necessarily attained in the interior. By the previous exercise, such minimum is 0. Conclusion: $f$ admits at least one zero in $\Delta$.

10. *Automorphisms of $\Delta$.*
    (a) *Show that every automorphism of the unit disc $\Delta$ extends injectively to a neighbourhood of its closure $\overline{\Delta}$ and admits at least a fixed point in $\overline{\Delta}$.*
    (b) *Show that if $f$ has a fixed point in $\Delta$, then it is necessarily unique.*

*Sol.*: Let $f(z) = e^{i\theta}\frac{z-z_0}{1-z\bar{z}_0}$ be an automorphism of the disc, where $\theta \in \mathbb{R}$ and $z_0 \in \Delta$. Since the rotation $z \mapsto e^{i\theta}z$ is defined and injective on all $\mathbb{C}$, it is sufficient to consider $g(z) = \frac{z-z_0}{1-z\bar{z}_0}$. The map $g$ is well defined provided that $1 - z\bar{z}_0 \neq 0$. This is true for all the $z$ with $|z| < 1/|z_0|$. Note that $1/|z_0| > 1$, meaning that $g$ is well defined on an open neighbourhood of $\overline{\Delta}$. To prove injectivity, we solve

$$w = \frac{z - z_0}{1 - z\bar{z}_0} \quad \Leftrightarrow \quad z = \frac{w + z_0}{1 + w\bar{z}_0}.$$

To see that there exists at least a fixed point in $\overline{\Delta}$, we consider the equation

$$e^{i\theta}\frac{z - z_0}{1 - z\bar{z}_0} = z \quad \Leftrightarrow \quad z^2 + z\,\frac{e^{i\theta} - 1}{\bar{z}_0} - e^{i\theta}\frac{z_0}{\bar{z}_0} = 0.$$

The equation shows that there are at most 2 fixed points. Since the modulus of their product is $|e^{i\theta}\frac{z_0}{\bar{z}_0}| = 1$, then either both fixed points are on $\partial\Delta$ or one is inside and the other is outside $\Delta$.

11. *Define $D(0, r) = \{z \in \mathbb{C} \ : \ |z| < r\}$. Let $f : D(0, 1) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \partial D(0, 1)$. Show that if $f$ is nonconstant, then there exists an automorphism $g$ of $D(0, 1)$ such that $f \circ g(0) = 0$.*

*Sol.*: By the maximum modulus principle, $|f(z)| < 1$, for all $z \in D(0, 1)$. In other words, $f$ maps the unit disc into itself. Since $|f|$ is constant on the boundary of the disc, then there exists $z_0 \in D(0, 1)$ such that $f(z_0) = 0$ (cf. Exercise 9). Since the automorphism group of the disc acts transitively on it, then there exists an automorphism $g$ such that $g(0) = z_0$. It follows that $f(g(0)) = f(z_0) = 0$.

12. *Let $f : \Delta \to \Delta$ be a holomorphic function with a zero of order $m$ at 0. Show that for all $z \in \Delta$ one has $|f(z)| \leq |z|^m$.*

*Sol.*: This exercise is a variation of Schwarz's Lemma. The power expansion of $f$ around 0 is $f(z) = a_m z^m + a_{m+1}z^{m+1} + \dots$. Consider the holomorphic function

$$g(z) := \begin{cases} \frac{f(z)}{z^m} = a_m + a_{m+1}z + \dots, & z \neq 0 \\ f^{(m)}(0), \quad z = 0. \end{cases}$$

For $0 < r < 1$, one has

$$|g(z)| \leq 1/r, \quad \forall z \ : \ |z| = r,$$

and by the maximum principle the above inequality holds for all $z$ with $|z| \leq r$. By letting $r \to 1$, we obtain $|g(z)| \leq 1$, for all $z \in \Delta$. Equivalently

$$|f(z)| \leq |z|^m,$$

as claimed.

13. *Verify that the Cayley transform*
$$C(z) := \; i\,\frac{1+z}{1-z}$$
*is a biholomorphism between the unit disc $\Delta$ and the upper half plane $\mathbb{H}^+$.*

*Sol.*: The map $C$ is holomorphic on $\Delta$. Since

$$Im\left(i\,\frac{1+z}{1-z}\right) = \frac{1-|z|^2}{|1-z^2|} > 0, \quad \forall z \in \Delta,$$

the map $C$ takes the disc onto the upperhalfplane. Finally $C$ is injective, as it has inverse

$$C^{-1}\colon \mathbb{H} \to \Delta, \quad w \mapsto \frac{w-i}{w+i}.$$

14. *Show that the strip $S_r := \{\, z \in \mathbb{C} \ : \ 0 < \mathrm{Im}\, z < r \,\}$, with $r > 0$, is biholomorphic to $\mathbb{H}$ (and therefore to $\Delta$). Determine an explicit biholomorphism $f\colon S_1 \to \Delta$.*

*Sol.*: The exponential map $e^z$ determines a biholomorphism between the strip $S_\pi$ and the upper-halfplane $\mathbb{H}$. It is holomorphic and injective:

$$e^z = e^w, \ z, w \in S_\pi \quad \Leftrightarrow \quad e^{z-w} = 1 \quad \Leftrightarrow \quad z - w \in \mathbb{Z}2\pi i \quad \Leftrightarrow \quad z = w.$$

It is surjective: horizontal lines in the strip $S_\pi$ are mapped into halflines from the origin in $\mathbb{H}$.
Any strip $S_r$, with $r > 0$, is biholomorphic to $S_\pi$ via the map $z \mapsto \frac{\pi}{r} z$.
An explicit biholomorphism $f\colon S_1 \to \Delta$ is given by

$$z \mapsto C^{-1}(\exp(\pi z)),$$

where $C^{-1}\colon \mathbb{H} \to \Delta$ is the inverse Cayley transform of Execise 13.
*Note*: The strip $S_r$ is a proper convex subset of $\mathbb{C}$. In particular it is simply connected. We'll see later in the course that any such set is biholomorphic to the unit disc.

15. *Determine whether there exists a nonconstant holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the border $\partial\Delta$ of the unit disc $\Delta$.*

*Sol.*: If $f(\mathbb{C})$ has empty intersection with the border $\partial\Delta$ of the unit disc $\Delta$, then either $f(\mathbb{C}) \subset \Delta$ or $f(\mathbb{C}) \subset \mathbb{C} \setminus \Delta$. In the first case $f$ is bounded, and therefore constant. In the second case $f(\mathbb{C})$ is contained in a domain $D$ biholomorphic to the unit disc, i.e. there exists a biholomorphism $g\colon D \to \Delta$ such that $g \circ f\colon \mathbb{C} \to \Delta$ is a bounded holomorphic function. By Liouville's theorem, $g \circ f$ is constant and therefore also $f$ is constant.

16. *Determine whether there exists a nonconstant holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$.*

*Sol.*: If $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$, then it is either contained in the upper halfplane or in the lower halfplane in $\mathbb{C}$. Both halfplanes are biholomorphic to the unit disc. Then the same argument as in the previous exercise shows that there are no nonconstant holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$.