1. Evaluate $\Gamma(-\frac{1}{2})$, $\Gamma(-\frac{7}{2})$, $\Gamma(-\frac{1}{3})$.

2. Let $n \in \mathbf{N}$. Show that

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2^n}\sqrt{\pi}\,.$$

3. Use Wielandt's theorem to prove that for $Re(z) > 0$

$$\Gamma(z) = \frac{1}{g(z)}, \quad \text{where} \quad g(z) = ze^{Cz}\prod_{k=1}^{\infty}\left((1+\frac{z}{k})e^{-z/k}\right),$$

where $C = \lim_k(1+\frac{1}{2}+\frac{1}{3}+\ldots-\log k)$ is the Euler-Mascheroni constant.
(*The definition of* Γ *as the reciprocal of the above function* g *is in Cartan's book, p.159-161*).

4. Use Wielandt's theorem to prove the Legendre duplication formula

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

(hint: let $w = 2z$ and consider the function $f(w) = 2^w\Gamma(\frac{w}{2})\Gamma(\frac{w}{2}+\frac{1}{2})$).

5. Prove that if $y > 0$, then

$$|\Gamma(iy)| = \sqrt{\frac{\pi}{y\sinh \pi y}}.$$

6. Recall that $\zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}}$, for $s \in \mathbf{C}$ with $Re(s) > 1$. Compute the logarithmic derivative of ζ (justify the steps).

7. Let 2, 3, 5, 7, ... be the series of prime numbers. Prove that

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{7^2})\ldots = \frac{6}{\pi^2}.$$

8. Verify that $\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s)$.

9. Using the analytic continuation given by the formula of the previous exercise, prove that $\zeta(-1) = -\frac{1}{12}$ and that $\zeta(-3) = \frac{1}{720}$.

10. Compute $\zeta(m)$, where m is a negative integer.