

1. (a) Show that \mathbf{C}^* and Δ^* are not biholomorphic;
 (b) Show that Δ^* is not biholomorphic to an annulus $A = \{z \in \mathbf{C} \mid 0 < r < |z| < 1\}$.

If you are curious, a proof that two annuli $A_{r,1} = \{0 < r < |z| < 1\}$ and $A_{s,1} = \{0 < s < |z| < 1\}$ are biholomorphic if and only if $r = s$ can be found in Greene-Krantz, Thm.7.9.1, p.240.

Infinite series of holomorphic and meromorphic functions.

2. Let D be a domain in \mathbf{C} . Prove that if the series of holomorphic functions $\sum_k g_k(z)$ converges normally on $A \subset D$, then it converges uniformly on $A \subset D$.
3. Show that the series $\sum_{n \geq 1} \frac{z^n}{n^2}$ converges uniformly on the unit disc.
4. (a) Show that $\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1}$, for $z \in \Delta$, and that $4 = \sum_k k \frac{1}{2^{k-1}}$.
 (b) Show that $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$, for $z \in \Delta$.
5. Show that $\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$ defines a meromorphic function on \mathbf{C} . Determine its poles and their orders.
6. Show that $\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ defines a meromorphic function on \mathbf{C} . Determine its poles and their orders.
7. Show that $g(z) = \left(\frac{\pi}{\sin \pi z}\right)^2$ converges uniformly to 0 for $|Im(z)| \rightarrow \infty$.
8. Compute $\lim_{z \rightarrow 0} \left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2}$ and $\lim_{z \rightarrow 0} \sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2}$. Deduce that $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.
9. Show that $\sum_{n \in \mathbf{Z}} \frac{1}{(z^3-n^3)}$ converges normally to a meromorphic function. Locate the poles and find the corresponding principal parts of the function.
10. Find a meromorphic function $f: \mathbf{C} \rightarrow \mathbf{C}$ with simple poles at the positive integers with residue 1.

Infinite products of holomorphic functions.

11. Prove that the infinite product $\prod_j a_j$, where $a_j = \frac{1}{2}$ for every j , does not converge.
12. Prove that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.
13. Prove that for $|z| < 1$ one has

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\dots = \frac{1}{1-z}.$$

14. Prove that

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

15. Construct an entire function on \mathbf{C} with simple zeros at n^2 , for $n \geq 0$, and no other zeros.
16. Construct an entire function that has simple zeros on the positive real axis at the points \sqrt{n} , for $n \geq 1$, double zeros on the imaginary axis at the points $\pm i\sqrt{n}$, for $n \geq 1$, and no other zeros.