*Uniform convergence on compact sets*

1. Show that the sequence $f_n(z) = \frac{1-z^{n+1}}{1-z}$ converges punctually to $\frac{1}{1-z}$, for $z \in \Delta$. Show that the convergence is locally uniform but not uniform on $\Delta$.

2. Consider the following sequences of functions
$$f_n(z) = \frac{z}{n}, \qquad g_n(z) = z^n, \qquad h_n(z) = \frac{1}{n^z}.$$
For each sequence:
(a) Find the pointwise limit;
(b) Find a set $A$ where the convergence is uniform;
(c) Find a set $U$ where the convergence is locally uniform;
(d) Determine whether the convergence is uniform on $U$ or not.

3. Let $\{p_n\}_{n\in\mathbf{N}}$ be a sequence of polynomials of degree $\leq N$, where $N$ is a fixed positive integer. Show that if $p_n \to p$ uniformly on compact sets, then the limit function $p$ is a polynomial of degree $\leq N$.

4. Let $D$ be a domain and let $f_n : D \to \mathbf{C}$ be a sequence of holomorphic functions converging to a *non-constant* function $f$ uniformly on compact sets. Show that if $f$ has $m$ zeros in $D$, then all but finitely many $f_n$ have at least $m$ zeros in $D$.

5. Let $g : D(0,\rho) \to \mathbf{C}$ be holomorphic. Let $G : D(0,\rho) \to \mathbf{C}$ be such that $G' = g$ and $G(0) = 0$. Show that for all $z \in D(0,\rho)$ one has
$$|G(z)| \leq |z| \sup\{|g(w)| \,:\, |w| \leq |z|\}.$$
Deduce that $\|G\|_{D(0,\rho)} \leq \rho \|G\|_{D(0,\rho)}$ (suggestion: take $G(z) := \int_{\gamma_z} g(w)dw$, where $\gamma_z(t) = tz$, for $t \in [0,1]$).

6. Let $f_n$ be a sequence of holomorphic functions converging locally uniformly to $f$ on $D(0,R)$. Show that $F_n \to F$ locally uniformly on $D(0,R)$, where $F_n' = f_n$, $F_n(0) = 0$ for all $n$, and $F' = f$, $F(0) = 0$.

7. Discuss the convergence and uniform convergence of the sequence $f_n(z) = nz^n$, for $n \in \mathbf{N}$.

8. Prove that the sequence $f_n(z) = \frac{1}{1+nz}$ is uniformly convergent to the function identically 0, for all $z$ such that $|z| \geq 2$. Can the region of uniform convergence be extended?

9. Let $\mathcal{F}$ be the family of all analytic functions on the open unit disc $\Delta$ whose coefficients in the Taylor expansion
$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots,$$
satisfy $|a_n| \leq n$, for each $n$. Show that $\mathcal{F}$ is relatively compact (with respect to the topology of locally uniform convergence).

10. Let $f_n$ be a sequence of analytic functions on $\Delta$, uniformly bounded. Assume that for each $z \in \Delta$ the sequence $f_n(z)$ converges. Show that $f_n$ converges uniformly on compact subsets of $\Delta$.

11. Consider the family $\mathcal{F} = \{f_n(z) = \frac{z}{n}\}_{n\in\mathbf{N}}$, defined on $\mathbf{C}$. Verify that $\mathcal{F}$ is not equibounded on $\mathbf{C}$, but it is equibounded on compact sets. Indeed it converges uniformly on compact sets to $f \equiv 0$.

*Some examples for comparing the real and the complex cases.*

a. (*Weierstrass' convergence theorem does not hold in the real case*) Consider the sequence of functions $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, defined on $[-1, 1]$. Verify that the functions $f_n$ are smooth, and converge uniformly to a non-smooth function $f$.

b. (*Hurwitz's theorem does not hold in the real case*) Consider the sequence $f_n : \mathbf{R} \to \mathbf{R}$, defined by $f_n(x) = x^2 + \frac{1}{n}$. Verify that the functions $f_n$ are never 0 on $\mathbf{R}$. Nevertheless they converge uniformly on compact sets to $f(x) = x^2$, which takes the value $f(0) = 0$.

c. (*Montel's theorem does not hold in the real case*) Consider the family $\mathcal{F} = \{f_n(x) = \sin(nx)\}_n$, defined on $[0, 2\pi]$. Verify that $\mathcal{F}$ is equibounded. Nevertheless it does not admit any converging subsequence (not even pointwise).