Laurent series and isolated singularities, Riemann extension theorem, Casorati-Weierstrass theorem, automorphisms of the plane, residue theorem, Rouche' theorem.

1. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions with $g(z) \neq 0$, for every $z \in \mathbb{C}$. Assume that $|f(z)| \leq |g(z)|$, for every $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ (cf. Exercise 17 in sheet 1). Show that the assumption "$g(z) \neq 0$, for every $z \in \mathbb{C}$" can be replaced with "g not identically zero".

2. (a) Let f be a bounded holomorphic function defined on $\mathbb{C} \setminus \{0, i, 1+i\}$ or on $C \setminus \mathbb{Z}$. Show that f is constant.
   (b) Is it true that f is constant when it is bounded on $\mathbb{C} \setminus (\{1/n : n \in \mathbb{N}\} \cup \{0\})$?

3. Show that the functions
$$\tfrac{\sin z}{z}, \qquad \tfrac{e^z - 1}{z}, \qquad \tfrac{\cosh z - 1}{z}$$
are entire, i.e. they extend holomorphically to $\mathbb{C}$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.
   (a) Show that if f has a zero of order n in z_0, then $1/f$ has a pole of order n in z_0.
   (b) Let z_0 be a singularity (removable, polar, essential) of f. Determine the corresponding type of singularity of $1/f$.

5. Let z_0 be a pole of f. Recall that for every $M >> 0$ there exists $\varepsilon > 0$ such that $f(D^*(z_0, \epsilon)) \subset \{|z| > M\}$.
   Show that, given $\varepsilon > 0$, there exists $M >> 0$ such that $\{|z| > M\} \subset f(D^*(z_0, \epsilon))$.
   (Justify your answer and mention all the results you used.)

6. Determine and classify all the singularities of the following functions:
$$\tan z, \quad \tfrac{1}{z^3} \textstyle\sum_{n=2}^{\infty} 2^n z^n, \quad z e^{1/z} e^{-1/z^2}, \quad \tfrac{\sin \frac{1}{z}}{z^4}, \quad \tfrac{\sin z}{z^4}, \quad \tfrac{1}{z^3} - \cos z.$$

7. Let $g(z) := e^{1/z} - e^{2/z}$. Determine $g(\mathbb{C} \setminus \{0\})$.

8. Let $f(z) = \frac{z^2}{z^2+1}$. Determine the Laurent serie of f around $z_0 = i$.

9. Let f be a holomorphic function on $D^*(0, r)$, $0 < r \leq +\infty$. Assume that $z_0 = 0$ is a pole of f of order m. Then the Laurent serie of f around z_0 is given by
$$f(z) = \textstyle\sum_{n \geq -m} a_n z^n, \qquad a_n = \tfrac{1}{(n+m)!} g^{(n+m)}(0), \quad \text{where } g(z) = z^m f(z).$$

10. Let $f(z) = \frac{e^z}{(z-i)^3 (z+2)^2}$.
   (i) Show that $z_1 = i$ is a pole of f of order 3.
   (ii) Determine all the coefficients of the principal part of the Laurent series of f around z_1.
   (iii) Show that $z_2 = -2$ is a pole of f of order 2.
   (iv) Determine all the coefficients of the principal part of the Laurent series of f around z_2.

11. Let $\gamma_2(\theta) = e^{i\theta}$, where $\theta \in [0, 2\pi]$, and $\gamma_1(\theta) = 3e^{i\theta}$, where $\theta \in [0, 2\pi]$. Compute

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2+5z}{(z-2)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2+5z}{(z-2)}dz;$$

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2-2}{z}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2-2}{z}dz;$$

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{z^3-3z-6}{z(z+2)(z+4)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^3-3z-6}{z(z+2)(z+4)}dz.$$

12. Compute the residues of the following functions at the singular points:

$$e^{\frac{3}{z^2}}, \qquad \frac{z^3}{z-1}, \qquad \frac{z^3}{(z-1)^2}, \qquad \frac{z^3}{1-z^4}, \qquad \frac{z^5}{(z^2-1)^2}, \qquad \frac{\cos z}{1+z+z^2}, \qquad \frac{1}{\sin z}.$$

13. Show that the function defined by $\sum_{n=1}^{\infty} \frac{1}{n! z^n}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Compute its integral on $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 2\pi]$.

14. Compute

$$\int_\gamma \frac{e^z}{z^3}dz, \qquad \int_\gamma \frac{e^{1/z}}{z^3}dz,$$

when $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 6\pi]$ and when $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 2\pi]$.

15. Use the "sector method" to compute

$$\int_0^{+\infty} \frac{1}{1+x^4}dx\,.$$

16. Let c be a complex number with $|c| > e$. Show that the equation $e^z = cz^n$ has n solutions in the unit disk.

17. Show that $z^5 + 15z + 1 = 0$ has 4 solutions in the annulus $L_{3/2,2}(0) = \{3/2 < |z| < 2\}$.