

Liouville's theorem, identity principle, maximum modulus principle, harmonic functions, Schwarz's lemma, automorphisms of the unit disk.

1. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic doubly periodic function (it means that $f(z + \omega_1) = f(z + \omega_2) = f(z)$ for all $z \in \mathbb{C}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are \mathbb{R} -linearly independent vectors). Then f is constant.
2. Let $U \subset \mathbb{C}$ be an open neighbourhood of 0. Show that there are no (non-constant) holomorphic functions $f: U \rightarrow \mathbb{C}$, such that:
 - (a) $f(\frac{1}{n}) = (-1)^n \frac{1}{n^2}$,
 - (b) $f(\frac{1}{n}) = \frac{1}{2^n}$,
 - (c) $|f^{(n)}(0)| > n!n^n$.
3. Let $D \subset \mathbb{C}$ be a domain and let $f: D \rightarrow \mathbb{C}$ be a holomorphic function, not identically zero. Prove that the set of zeros of f in D is at most countable (use: D is a countable union of compact sets).
4. Let $f: U \rightarrow \mathbb{C}$ be a holomorphic function, where U is an open neighbourhood of the closed unit disc $\overline{\Delta}$. Assume that f is not identically zero.
 - (a) Show that f has at most finitely many zeros in Δ .
 - (b) Determine the zeros of $f(z) = \sin(\frac{1}{1-z})$ on the disc Δ and compare the result with (a).
5. Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \frac{\pi}{2}\}$. Determine whether there exists a holomorphic function $f: S \rightarrow \mathbb{C}$ such that
 - (a) $\operatorname{Re} f(z) = x^2y + y^2x + \sin x \sinh y$;
 - (b) $\operatorname{Re} f(z) = y^3 - 3x^2y + \cos x \cosh y + \sin x \sinh y$.
 Justify your answer: either exhibit one such function or explain why it cannot exist.
6. (*Liouville's theorem for harmonic functions*). Let $u: \mathbb{C} \rightarrow \mathbb{R}$ be harmonic and bounded either from above or from below.
 - (a) Show that u is constant.
 - (b) Verify that the real and the imaginary parts of the following holomorphic functions are not bounded:

$$e^z, \quad \sin z, \quad \cos z, \quad z^2.$$

7. Set $D = D(z_0, r)$. Let $f: D \rightarrow \mathbb{C}$ be a holomorphic function. Show that

$$f(z_0) = \frac{1}{\operatorname{Area}(D)} \int_D f(z) dx dy.$$

8. Let D be a domain in \mathbb{C} and let $f: D \rightarrow \mathbb{C}$ be a nonconstant holomorphic function. Show that the local minima of $|f|$ coincide with the zeros of f .
9. Let $f: U \rightarrow \mathbb{C}$ be a nonconstant holomorphic function defined on a neighbourhood of the unit disc Δ . Show that if $|f|$ is constant on the boundary of Δ , then f admits at least one zero in Δ .

10. *Automorphisms of Δ .*

- (a) Show that every automorphism of the unit disc Δ extends injectively to a neighbourhood of its closure $\overline{\Delta}$ and admits at least a fixed point in $\overline{\Delta}$.
- (b) Show that if f has a fixed point in Δ , then it is necessarily unique.

11. Define $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$. Let $f : D(0, 1) \rightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \partial D(0, 1)$. Show that if f is nonconstant, then there exists an automorphism g of $D(0, 1)$ such that $f \circ g(0) = 0$.

12. Let $f : \Delta \rightarrow \Delta$ be a holomorphic function with a zero of order m at 0. Show that for all $z \in \Delta$ one has $|f(z)| \leq |z|^m$.

13. Verify that the Cayley transform

$$C(z) := i \frac{1+z}{1-z}$$

is a biholomorphism between the unit disc Δ and the upper half plane \mathbb{H}^+ .

14. Show that the strip $S_r := \{z \in \mathbb{C} : 0 < \text{Im } z < r\}$, with $r > 0$, is biholomorphic to \mathbb{H}^+ (and therefore to Δ). Determine an explicit biholomorphism $f: S_1 \rightarrow \Delta$.

15. Determine whether there exists a nonconstant holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the border $\partial\Delta$ of the unit disc Δ .

16. Determine whether there exists a nonconstant holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the real line \mathbb{R} .